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# PROBLEMS ON COMBINATORIAL PROPERTIES OF PRIMES

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**ABSTRACT.** For  $x \geq 0$  let  $\pi(x)$  be the number of primes not exceeding  $x$ . The asymptotic behaviors of the prime-counting function  $\pi(x)$  and the  $n$ -th prime  $p_n$  have been studied intensively in analytic number theory. Surprisingly, we find that  $\pi(x)$  and  $p_n$  have many combinatorial properties which should not be ignored. In this paper we pose 60 open problems on combinatorial properties of primes for further research. For example, we conjecture that for any integer  $n > 1$  one of the  $n$  numbers  $\pi(n), \pi(2n), \dots, \pi(n^2)$  is prime; we also conjecture that for each  $n = 1, 2, 3, \dots$  there is a number  $k \in \{1, \dots, n\}$  such that the number of twin prime pairs not exceeding  $kn$  is a square.

## 1. INTRODUCTION

Prime numbers play important roles in number theory. For  $x \geq 0$  let  $\pi(x)$  denote the number of primes not exceeding  $x$ . The celebrated Prime Number Theorem states that

$$\pi(x) \sim \text{Li}(x) \quad \text{as } x \rightarrow +\infty,$$

where  $\text{Li}(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$ . This has the following equivalent version:

$$p_n \sim n \log n \quad \text{as } n \rightarrow +\infty,$$

where  $p_n$  denotes the  $n$ -th prime. To get sharp estimation for  $\pi(x)$  is a main research topic in analytic number theory. It is known (cf. [Sc]) that Riemann's Hypothesis implies that

$$\pi(x) = \text{Li}(x) + O(\sqrt{x} \log x) \quad \text{and} \quad p_{n+1} - p_n = O(\sqrt{p_n} \log p_n).$$

Many number theorists consider primes generally irregular and only focus on the asymptotic behaviour of primes. In contrast with the great achievements

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on the asymptotic behaviors of  $\pi(x)$  and  $p_n$  (see [Z] for a recent breakthrough), almost nobody has investigated combinatorial properties of primes seriously and systematically.

Surprisingly, we find that the functions  $\pi(x)$  and  $p_n$  have many unexpected combinatorial properties depending on their exact values. Also, partition functions arising from combinatorics have nice connections with primes. In this paper we pose 60 typical conjectures in this direction which might interest number theorists and stimulate further research. The next section contains 25 conjecture on combinatorial properties of  $\pi(x)$ , while Section 3 contains 25 conjectures on combinatorial properties of the function  $p_n$ . Section 4 is devoted to 10 conjectures on primes related to partition functions. The reader may also consult [S13a] for the author's previous conjectures on alternating sums of consecutive primes.

## 2. COMBINATORIAL PROPERTIES OF $\pi(x)$ AND RELATED THINGS

**Conjecture 2.1** (2014-02-09). (i) *For any integer  $n > 1$ ,  $\pi(kn)$  is prime for some  $k = 1, \dots, n$ . Moreover, for every  $n = 1, 2, 3, \dots$ , there is a positive integer  $k < 3\sqrt{n} + 3$  with  $\pi(kn)$  prime.*

(ii) *For each  $n = 6, 7, \dots$ , there is a positive integer  $k < n$  such that  $k^2 + k - 1$  and  $\pi(kn)$  are both prime. For any integer  $n > 3$ , there is a positive integer  $k < n$  such that  $k^2 + k - 1$  and  $\pi(kn) + 1$  are both prime. Also, for any integer  $n > 6$ , there is a positive integer  $k < n$  such that  $k^2 + k - 1$  and  $\pi(kn) - 1$  are both prime.*

*Remark 2.1.* We also conjecture that for each  $n = 2, 3, \dots$  there is a prime  $p \leq p_n$  with  $\pi(pn)$  prime. We have verified part (i) of the conjecture for  $n$  up to  $10^6$ . See [S14, A237578] for the sequence  $a(n) = |\{0 < k < n : \pi(kn) \text{ is prime}\}|$  ( $n = 1, 2, 3, \dots$ ). See also [S14, A237615] for part (ii) of the conjecture.

**Conjecture 2.2** (2014-02-10). (i) *For any positive integer  $n$ , there is a positive integer  $k < p_n$  such that  $\pi(kn) \equiv 0 \pmod{n}$ .*

(ii) *For each positive integer  $n$ , the set  $\{\pi(kn) : k = 1, \dots, 2p_n\}$  contains a complete system of residues modulo  $n$ .*

*Remark 2.2.* See [S14, A237598] and [S14, A237643] for related sequences concerning this conjecture.

**Conjecture 2.3** (2014-02-20). *Let  $n > 1$  be an integer. Then  $\pi(jn) \mid \pi(kn)$  for some  $1 \leq j < k \leq n$  with  $k \equiv 1 \pmod{j}$ .*

*Remark 2.3.* For example,  $\pi(3 \times 50) = 35$  divides  $\pi(7 \times 50) = 70$ . We have verified the conjecture for all  $n = 2, 3, \dots, 21500$ . See [S14, A238224] for a related sequence.

**Conjecture 2.4.** (i) (2014-02-10) *For any positive integer  $n$ , there is a positive integer  $k < p_n$  such that  $\pi(kn)$  is a square.*

(ii) (2014-02-14) *Let  $n$  be any positive integer. Then, for some  $k = 1, \dots, n$ , the number of twin prime pairs not exceeding  $kn$  is a square.*

*Remark 2.4.* Similar to part (i), we conjecture that for any integer  $n > 9$  there is a positive integer  $k < p_n/2$  such that  $\pi(kn)$  is a triangular number. We have verified part (ii) of the conjecture for all  $n = 1, \dots, 21580$ ; for example, for  $n = 19939$  we may take  $k = 12660$  since there are exactly  $1000^2 = 10^6$  twin prime pairs not exceeding  $12660 \times 19939 = 252427740$ . See [S14, A237840, A237879, A237975] for some related sequences.

**Conjecture 2.5** (2014-02-24). *For any integer  $n > 5$ , there is a positive integer  $k < n$  with  $kn + \pi(kn)$  prime.*

*Remark 2.5.* This implies that there are infinitely many positive integers  $m$  with  $m + \pi(m)$  prime. See [S14, A237712] for a related sequence.

**Conjecture 2.6** (2014-02-17). *For any integer  $n > 4$  and  $k = 1, \dots, n$ , we have  $\pi(kn)^{1/k} > \pi((k+1)n)^{1/(k+1)}$ .*

*Remark 2.6.* Recall that a conjecture of Firoozbakht (cf. [R, p.185]) asserts that the sequence  $p_n^{1/n}$  ( $n = 1, 2, 3, \dots$ ) is strictly decreasing. The paper [S13b] contains many conjectures on monotonicity of arithmetical sequences.

**Conjecture 2.7** (2014-02-22). *Let  $n$  be any positive integer. Then,  $\pi(kn + n) - \pi(kn)$  (the number of primes in the interval  $(kn, (k+1)n]$ ) is a square for some  $k = 0, \dots, n-1$ .*

*Remark 2.7.* See [S14, A238277] for a related sequence.

**Conjecture 2.8** (2014-02-22). *Let  $n > 3$  be an integer. Then, for some  $k = 1, \dots, n-1$ , both  $\pi(kn) - \pi((k-1)n)$  and  $\pi((k+1)n) - \pi(kn)$  are prime.*

*Remark 2.8.* See [S14, A238278] for a related sequence.

**Conjecture 2.9** (2014-02-22). (i) *For any integer  $n > 1$ , there is a positive integer  $k < n$  such that the intervals  $(kn, (k+1)n)$  and  $((k+1)n, (k+2)n)$  contain the same number of primes, i.e.,*

$$\pi(kn), \pi((k+1)n), \pi((k+2)n)$$

*form a 3-term arithmetic progression.*

(ii) *For any integer  $n > 4$ , there is a positive integer  $k < p_n$  such that*

$$\pi(kn), \pi((k+1)n), \pi((k+2)n), \pi((k+3)n)$$

*form a 4-term arithmetic progression.*

*Remark 2.9.* See [S14, A238281] for a sequence related to part (i) of this conjecture.

**Conjecture 2.10** (2014-02-23). *For any positive integer  $n$ , we have*

$$|\{\pi((k+1)n) - \pi(kn) : k = 0, \dots, n-1\}| \geq \sqrt{n-1},$$

*and equality holds only when  $n$  is 2 or 26.*

*Remark 2.10.* See [S14, A230022] for a related sequence.

**Conjecture 2.11** (2014-02-24). *Let  $n > 1$  be an integer. Then, for some prime  $p \leq p_n$ , the three numbers  $\pi(p), \pi(p+n), \pi(p+2n)$  form a nontrivial arithmetic progression, i.e.,  $\pi(p+2n) - \pi(p+n) = \pi(p+n) - \pi(p) > 0$ .*

*Remark 2.11.* See [S14, A210210] for a related sequence.

**Conjecture 2.12** (2014-02-08). (i) *Any integer  $n > 5$  can be written as  $k+m$  with  $k$  and  $m$  positive integers such that  $\pi(k) + \pi(m) - 2$  is prime.*

(ii) *Each integer  $n > 10$  can be written as  $k+m$  with  $k$  and  $m$  positive integers such that  $\pi(km)$  (or  $\pi(k^2m)$ ) is prime.*

*Remark 2.12.* See [S14, A237496 and A237497] for related sequences. Similar to part (ii), we also conjecture (cf. [S14, A237531]) that for any integer  $n > 5$  there is a positive integer  $k < n/2$  such that  $\varphi(k(n-k)) - 1$  and  $\varphi(k(n-k)) + 1$  are twin prime, where  $\varphi$  denotes Euler's totient function.

For  $x \geq 0$ , we use  $\pi_2(x)$  to denote the number of twin prime pairs not exceeding  $x$ .

**Conjecture 2.13** (2014-02-15). (i) *Any integer  $n > 7$  can be written as  $k+m$  with  $k$  and  $m$  positive integers such that  $\pi_2(k) + \pi_2(m)$  is prime.*

(ii) *Each integer  $n > 5$  can be written as  $k+m$  with  $k$  and  $m$  positive integers such that  $\pi_2(km)$  is prime. Also, any integer  $n > 8$  can be written as  $k+m$  with  $k$  and  $m$  positive integers such that  $\pi_2(km) - 1$  and  $\pi_2(km) + 1$  are twin prime.*

*Remark 2.13.* This is an analogue of Conjecture 2.12 for twin prime pairs.

**Conjecture 2.14.** (i) (2014-02-11) *For any positive integer  $n$ , the set  $\{\pi(k^2) : k = 1, \dots, 2p_{n+1} - 3\}$  contains a complete system of residues modulo  $m$ .*

(ii) (2014-02-17) *The sequence  $\sqrt[n]{\pi(n^2)}$  ( $n = 3, 4, \dots$ ) is strictly decreasing.*

(iii) (2014-02-17) *Let  $n > 0$  be an integer. Then the interval  $[\pi(n^2), \pi((n+1)^2)]$  contains at least one prime except for  $n = 25, 35, 44, 46, 105$ .*

*Remark 2.14.* Legendre's conjecture asserts that for each positive integer  $n$  there is a prime between  $n^2$  and  $(n+1)^2$ .

**Conjecture 2.15** (2014-02-11). (i) *For any integer  $n > 8$ ,  $\pi(k)$  and  $\pi(k^2)$  are both prime for some integer  $k \in (n, 2n)$ .*

(ii) *There are infinitely many primes  $p$  with  $\pi(p)$ ,  $\pi(\pi(p))$  and  $\pi(p^2)$  all prime.*

*Remark 2.15.* See [S14, A237657 and A237687] for related sequences and data.

**Conjecture 2.16.** (i) (2014-02-09) *For any integer  $n > 1$ ,  $\pi(n + k^2)$  is prime for some  $k = 1, \dots, n - 1$ . In general, for each  $a = 2, 3, \dots$ , if an integer  $n$  is sufficiently large, then  $\pi(n + k^a)$  is prime for some  $k = 1, \dots, n - 1$ .*

(ii) (2014-02-10) *Let  $n > 4$  be an integer. Then  $n + \pi(k^2)$  is prime for some  $k = 1, \dots, n$ .*

(iii) (2014-02-10) *For every  $n = 2, 3, \dots$  there is a positive integer  $k < n$  with  $p = n - \pi(k(k + 1)/2)$  prime.*

*Remark 2.16.* See [S14, A237582 and A237595] for related sequences. The author ever conjectured that if  $n$  is a positive integer then  $n + k$  and  $n + k^2$  are both prime for some  $k = 0, \dots, n$  (cf. [S14, A185636]).

**Conjecture 2.17** (2014-02-08). (i) *For any integer  $n > 4$ , there is a prime  $p < n$  with  $pn + \pi(p)$  prime. Moreover, for every positive integer  $n$ , there is a prime  $p < \sqrt{2n} \log(5n)$  with  $pn + \pi(p)$  prime.*

(ii) *For any integer  $n > 8$ , there is a prime  $p \leq n + 1$  such that  $(p - 1)n - \pi(p - 1)$  is prime.*

*Remark 2.17.* We have verified part (i) for all  $n = 5, 6, \dots, 10^8$ ; see [S14, A237453] for a related sequence. We also conjecture that for every  $n = 1, 2, 3, \dots$  there is a positive integer  $k < 3\sqrt{n}$  such that  $kn + p_k$  is prime.

**Conjecture 2.18** (2013-11-24). (i) *Every  $n = 4, 5, \dots$  can be written as  $p + q - \pi(q)$ , where  $p$  and  $q$  are odd primes not exceeding  $n$ .*

(ii) *For any integer  $n > 7$ , there is a prime  $p < n$  such that  $n + p - \pi(p)$  is prime.*

*Remark 2.18.* For example,  $9 = 7 + 5 - \pi(5)$  with 7 and 5 prime. See [S14, A232463 and A232443] for related sequences.

**Conjecture 2.19** (2014-02-06). (i) *For any integer  $n > 2$ , there is a prime  $p < 2n$  with  $\pi(p)$  and  $2n - p$  both prime.*

(ii) *For any integer  $n > 36$ , we can write  $2n - 1$  as a sum of three elements of the set  $\{p : p \text{ and } \pi(p) \text{ are both prime}\}$ .*

*Remark 2.19.* Part (i) is a refinement of Goldbach's conjecture, and part (ii) is stronger than the weak Goldbach conjecture finally proved by H. A. Helfgott [He]. See [S14, A237284 and A237291] for related representation functions.

Recall that a prime  $p$  with  $2p + 1$  also prime is called a Sophie Germain prime.

**Conjecture 2.20** (2014-02-13). (i) *For any integer  $n > 4$ , there is a prime  $p < n$  such that  $\pi(n - p)$  is a Sophie Germain prime. Also, for any integer  $n > 8$  there is a prime  $p < n$  such that  $\pi(n - p) - 1$  and  $\pi(n - p) + 1$  are twin prime.*

(ii) *For any integer  $n > 4$ , there is a prime  $p < n$  such that  $3m \pm 1$  and  $3m + 5$  are all prime with  $m = \pi(n - p)$ . Also, for any integer  $n > 8$ , there is a prime  $p < n$  such that  $3m \pm 1$  and  $3m - 5$  are all prime with  $m = \pi(n - p)$ .*

*Remark 2.20.* See [S14, A237768 and A237769] for related sequences. We have verified the second assertion in part (i) for  $n$  up to  $3 \times 10^7$ .

**Conjecture 2.21** (2014-02-13). (i) *For any integer  $n > 4$ , there is a prime  $p < n$  such that the number of Sophie Germain primes among  $1, \dots, n - p$  is a Sophie Germain prime.*

(ii) *For any integer  $n > 12$ , there is a prime  $p < n$  such that*

$$r = |\{q \leq n - p : q \text{ and } q + 2 \text{ are twin prime}\}|$$

*and  $r + 2$  are twin prime.*

*Remark 2.21.* See [S14, A237815 and A237817] for related sequences.

**Conjecture 2.22** (2014-02-12). (i) *For any integer  $n > 2$ , there is a prime  $p < n$  such that  $\pi(n - p)$  is a square. Also, for any integer  $n > 2$  there is a prime  $p < n$  such that  $\pi(n - p)$  is a triangular number.*

(ii) *For any integer  $n \geq 10900$ , there is a prime  $p < n$  such that  $\pi(n - p)$  is a cube.*

(iii) *For any integer  $n > 2$ , there is a prime  $p \leq p_n$  such that  $\pi(n + p)$  is a square.*

*Remark 2.22.* We have verified the first assertion in part (i) for  $n$  up to  $5 \times 10^8$ . See [S14, A237706 and A237710] for related sequences.

**Conjecture 2.23** (2014-02-13). (i) *For any integer  $n > 11$ , there is a prime  $p < n$  such that the number of Sophie Germain primes among  $1, \dots, n - p$  is a square.*

(ii) *For any integer  $n \geq 54$ , there is a prime  $p < n$  such that the number of Sophie Germain primes among  $1, \dots, n - p$  is a cube.*

*Remark 2.23.* This is an analogue of Conjecture 2.20 for Sophie Germain primes. See [S14, A237837] for a sequence related to part (ii).

**Conjecture 2.24.** (i) (2014-02-13) *For any integer  $n > 5$ , there is a prime  $p < n$  such that  $\lfloor \sqrt{n - p} \rfloor$  is a Sophie Germain prime.*

(ii) (2014-02-14) *For any integer  $n > 1$ , there is a number  $k \in \{1, \dots, n\}$  such that the number of Sophie Germain primes not exceeding  $kn$  is a Sophie Germain prime.*

*Remark 2.24.* Note that  $\lfloor \sqrt{n-p} \rfloor$  is the number of squares among  $1, \dots, n-p$ . See [S14, A237819 and A237838] for related sequences.

**Conjecture 2.25** (2014-02-22). *For any integer  $n > 4$ , there is a number  $k \in \{1, \dots, n\}$  such that the number of prime ideals of the Gaussian ring  $\mathbb{Z}[i]$  with norm not exceeding  $kn$  is a prime congruent to 1 modulo 4.*

*Remark 2.25.*  $\mathbb{Z}[i]$  is a principal ideal domain, and any prime ideal  $P$  of it has the form  $(p)$  with  $p$  a rational prime congruent to 3 modulo 4 or  $p = a + bi$  with  $N(p) = a^2 + b^2$  a rational prime not congruent to 3 modulo 4. So, the number of prime ideals of  $\mathbb{Z}[i]$  with norm not exceeding  $x$  actually equals

$$\pi(\sqrt{x}) + |\{\sqrt{x} < p \leq x : p \text{ is a prime with } p \not\equiv 3 \pmod{4}\}|.$$

### 3. COMBINATORIAL PROPERTIES INVOLVING THE FUNCTION $p_n$

**Conjecture 3.1** (Unification of Goldbach's Conjecture and the Twin Prime Conjecture, 2014-01-29). *For any integer  $n > 2$ , there is a prime  $q$  with  $2n - q$  and  $p_{q+2} + 2$  both prime.*

*Remark 3.1.* We have verified this for  $n$  up to  $2 \times 10^8$ . See [S14, A236566] for a related sequence. Note that the conjecture implies the twin prime conjecture. In fact, if all primes  $q$  with  $p_{q+2} + 2$  prime are smaller than an even number  $N > 2$ , then for any such a prime  $q$  the number  $N! - q$  is composite since  $N! - q \equiv 0 \pmod{q}$  and  $N! - q \geq q(q+1) - q > q$ .

**Conjecture 3.2** (Super Twin Prime Conjecture, 2014-02-05). *Any integer  $n > 2$  can be written as  $k + m$  with  $k$  and  $m$  positive integers such that  $p_k + 2$  and  $p_{p_m} + 2$  are both prime.*

*Remark 3.2.* We have verified the conjecture for  $n$  up to  $10^9$ . See [S14, A218829, A237259, A237260] for related sequences. If  $p, p+2$  and  $\pi(p)$  are all prime, then we call  $\{p, p+2\}$  a super twin prime pair. Conjecture 3.1 implies that there are infinitely many super twin prime pairs. In fact, if all those positive integer  $m$  with  $p_{p_m} + 2$  prime are smaller than an integer  $N > 2$ , then by Conjecture 3.1, for each  $j = 1, 2, 3, \dots$ , there are positive integers  $k(j)$  and  $m(j)$  with  $k(j) + m(j) = jN$  such that  $p_{k(j)} + 2$  and  $p_{p_{m(j)}} + 2$  are both prime, and hence  $k(j) \in ((j-1)N, jN)$  since  $m(j) < N$ ; thus

$$\sum_{j=1}^{\infty} \frac{1}{p_{k(j)}} \geq \sum_{j=1}^{\infty} \frac{1}{p_{jN}},$$

which is impossible since the series on the right-hand side diverges while the series on the left-hand side converges by Brun's theorem on twin primes.

**Conjecture 3.3** (2014-01-28). *Any integer  $n > 2$  can be written as  $k + m$  with  $k$  and  $m$  positive integers such that both  $\{6k \pm 1\}$  and  $\{p_m, p_m + 2\}$  are twin prime pairs.*

*Remark 3.3.* Clearly this implies the twin prime conjecture. We have verified Conjecture 3.3 for  $n$  up to  $2 \times 10^7$ . See [S14, A236531] for related data and graphs.

**Conjecture 3.4** (2014-02-07). *Any integer  $n > 1$  can be written as  $k + m$  with  $k$  and  $m$  positive integers such that  $p_k^2 - 2$ ,  $p_m^2 - 2$  and  $p_{p_m}^2 - 2$  are both prime.*

*Remark 3.3.* We have verified this for  $n$  up to  $10^8$ . See [S14, A237413 and A237414] for related sequences. It is not yet proven that there are infinitely many primes of the form  $x^2 - 2$  with  $x \in \mathbb{Z}$ .

**Conjecture 3.5** (2013-11-25). *The set*

$$\{k + m : 0 < k < m < n, \text{ and } p_k, p_m, p_n \text{ form an arithmetic progression}\}$$

*coincides with  $\{5, 6, 7, \dots\}$ .*

*Remark 3.5.* It is known that there are infinitely many three-term arithmetic progressions of primes. See [S14, A232502] for a sequence related to the conjecture.

**Conjecture 3.6** (2014-02-22). *For every  $n = 1, 2, 3, \dots$ , there is a positive integer  $k \leq 3p_n + 8$  such that  $p_{kn}, p_{(k+1)n}, p_{(k+2)n}$  form a 3-term arithmetic progression.*

*Remark 3.4.* Recall that the Green-Tao theorem (cf. [GT]) asserts that there are arbitrarily long arithmetic progressions of primes. See [S14, A238289] for a sequence related to the conjecture.

**Conjecture 3.7.** (i) (2013-12-01) *There are infinitely many positive integers  $n$  such that*

$$n \pm 1, p_n \pm n, np_n \pm 1$$

*are all prime.*

(ii) (2014-01-20) *There are infinitely many primes  $q$  with  $p_q^2 + 4q^2$  and  $q^2 + 4p_q^2$  both prime.*

*Remark 3.7.* For part (i), the first such a number  $n$  is 22110; see [S14, A232861] for a list of the first 2000 such numbers  $n$ . See also [S14, A236193] for a list of the first 10000 suitable primes  $q$  in part (ii).



**Conjecture 3.8** (2013-11-23). *Let  $n > 6$  be an integer. Then  $p_k + p_{n-k} - 1$  is prime for some  $k = 1, \dots, n-1$ . Also,  $p_k^2 + p_{n-k}^2 - 1$  is prime for some  $k = 1, \dots, n-1$ .*

*Remark 3.8.* See [S14, A232465] for a related sequence.

**Conjecture 3.9** (2013-12-07). *For every  $n = 2, 3, \dots$ , there is a positive integer  $k < n$  with  $kp_{n-k} + 1$  prime. Also, for any integer  $n > 2$ , there is a positive integer  $k < n$  with  $kp_{n-k} - 1$  prime.*

*Remark 3.9.* See [S14, A233296] for a related sequence.

**Conjecture 3.10** (2013-12-11). *Let  $n > 5$  be an integer. Then  $p_k p_{n-k} - 6$  is prime for some  $0 < k < n$ .*

*Remark 3.10.* See [S14, A233529] for the sequence  $a(n) = |\{0 < k < n : p_k p_{n-k} - 6 \text{ is prime}\}|$  ( $n = 1, 2, 3, \dots$ ). By J.-R. Chen's work [C], there are infinitely many primes  $p$  with  $p - 6$  a product of at most two primes.

**Conjecture 3.11** (2013-12-10). (i) *For any integer  $n > 3$ , there is a positive integer  $k < n$  such that  $p_k^2 + 4p_{n-k}^2$  is prime.*

(ii) *Let  $n > 10$  be an integer. Then there is a positive integer  $k < n$  with  $p_k^3 + 2p_{n-k}^3$  prime. Also,  $p_k^3 + 2p_{n-k}^2$  is prime for some  $0 < k < n$ .*

*Remark 3.11.* See [S14, A233439] for a sequence related to part (i). In 2001 Heath-Brown [HB] proved that there are infinitely many primes of the form  $x^3 + 2y^3$  with  $x, y \in \mathbb{Z}$ .

**Conjecture 3.12** (2013-12-05). (i) *Any integer  $n > 7$  can be written as  $k + m$  with  $k$  and  $m$  positive integers such that  $2^k + p_m$  is prime.*

(ii) *Each integer  $n > 3$  can be written as  $k + m$  with  $0 < k < m$  such that  $2^k p_m + 3$  is prime.*

*Remark 3.12.* See [S14, A233150 and A233204] for related sequences. We have verified part (i) of the conjecture for  $n$  up to  $3 \times 10^7$ ; for example, for  $n = 28117716$  we may take  $k = 81539$  and  $m = 28036177$ . Part (i) was motivated by the author's conjecture (cf. [S13c] and [S14, A231201]) that any integer  $n > 1$  can be written as a sum of two positive integers  $k$  and  $m$  such that  $2^k + m$  is prime.

**Conjecture 3.13** (2013-12-05). *Any integer  $n > 2$  can be written as  $k + m$  with  $k$  and  $m$  positive integers such that  $\binom{2k}{k} + p_m$  is prime.*

*Remark 3.13.* For example,  $9 = 2 + 7$  with  $\binom{2 \cdot 2}{2} + p_7 = 6 + 17 = 23$  prime. We have verified the conjecture for  $n$  up to  $10^8$ . See [S14, A233183] for related data and graphs.

**Conjecture 3.14** (2013-12-06). *Any integer  $n > 3$  can be written as  $k + m$  with  $k$  and  $m$  positive integers such that  $k! + p_m$  is prime.*

*Remark 3.14.* For example,  $11 = 4 + 7$  with  $4! + p_7 = 24 + 17 = 41$  prime. We have verified the conjecture for  $n$  up to  $10^7$ . See [S14, A233206] for related data and graphs.

**Conjecture 3.15** (2014-01-21). *For any integer  $n \geq 20$ , there is a positive integer  $k < n$  such that  $m = \varphi(k) + \varphi(n - k)/8$  is an integer with  $\binom{2m}{m} + p_m$  prime.*

*Remark 3.15.* This implies that there are infinitely many positive integers  $m$  with  $\binom{2m}{m} + p_m$  prime. (By Stirling's formula,  $\binom{2m}{m} \sim 4^m / \sqrt{m\pi}$  as  $m \rightarrow \infty$ .) See [S14, A236241] for the corresponding representation function, and [S14, A236242] for a list of 52 values of  $m$  with  $\binom{2m}{m} + p_m$  prime. For example, when  $m = 30734$  the number  $\binom{2m}{m} + p_m$  is a prime with 18502 decimal digits.

**Conjecture 3.16** (2013-12-29). *Any integer  $n > 9$  can be written as  $k + m$  with  $k$  and  $m$  positive integers such that  $q = k + p_m$  and  $p_q - q + 1$  are both prime.*

*Remark 3.16.* For example,  $4 = 10 + 4$  with  $10 + p_4 = 17$  and  $p_{17} - 16 = 59 - 16 = 43$  both prime. See [S14, A234694] for related data and graphs, and [S14, A234695] for the first 10000 primes  $q$  with  $p_q - q + 1$  also prime.

**Conjecture 3.17** (2014-01-04). *For any integer  $n > 6$ , there is a prime  $q < n/2$  with  $p_q - q + 1$  also prime such that  $n - (1 + \{n\}_2)q$  is prime, where  $\{n\}_2$  is 0 or 1 according as  $n$  is even or odd.*

*Remark 3.17.* See [S14, A235189] for the representation function. The conjecture is stronger than both Goldbach's conjecture and Lemoine's conjecture that any odd number  $n > 5$  can be written as  $p + 2q$  with  $p$  and  $q$  both prime.

**Conjecture 3.18** (2014-01-30). *Any odd number greater than 5 can be written as a sum of three elements of the set*

$$\{q : \text{both } q \text{ and } p_q - q + 1 \text{ are prime}\}.$$

*Remark 3.18.* This is stronger than the weak Goldbach conjecture finally proved by Helfgott [He]. See [S14, 236832] for the corresponding representation function.

**Conjecture 3.19** (2014-01-19). *For any integer  $n \geq 32$ , there is a positive integer  $k < n - 2$  such that  $q = \varphi(k) + \varphi(n - k)/2 + 1$  and  $p_q - q \pm 1$  are all prime.*

*Remark 3.19.* See [S14, A236097 and A236119] for related sequences. The conjecture implies that there are infinitely twin prime pairs of the form  $p_q - q \pm 1$  with  $q$  prime.

**Conjecture 3.20** (2014-01-17). *For any integer  $n \geq 38$ , there is a positive integer  $k < n$  such that  $q = \varphi(k) + \varphi(n - k)/3 + 1$ ,  $r = p_q - q + 1$  and  $s = p_r - r + 1$  are all prime.*

*Remark 3.20.* See [S14, A235924 and A235925] for related sequences. The conjecture implies that there are infinitely primes  $q$  with  $r = p_q - q + 1$  and  $s = p_r - r + 1$  both prime.

**Conjecture 3.21** (2014-01-17). *For each  $m = 2, 3, \dots$ , there is a prime chain  $q_1 < \dots < q_m$  of length  $m$  such that  $q_{k+1} = p_{q_k} - q_k + 1$  for all  $0 < k < m$ .*

*Remark 3.21.* For such chains of length  $m = 4, 5, 6$ , see [S14, A235934, A235935 and A235984].

**Conjecture 3.22** (2014-01-04). *Any integer  $n > 8$  can be written as  $k(k + 1)/2 + m$  with  $k$  and  $m$  positive integers such that  $p_{k(k+1)/2} + \varphi(m)$  is prime.*

*Remark 3.22.* Note that those  $k(k + 1)/2$  ( $k = 1, 2, \dots$ ) are triangular numbers.

**Conjecture 3.23** (2013-12-16). (i) *Any integer  $n > 100$  can be written  $k^2 + m$  with  $k$  and  $m$  positive integers such that  $\varphi(k^2) + p_m$  is prime.*

(ii) *If an integer  $n > 6$  is not equal to 18, then it can be written as  $k^2 + m$  with  $k$  and  $m$  positive integers such that  $\sigma(k^2) + p_m - 1$  is prime.*

*Remark 3.23.* See [S14, A236548] for a sequence related to part (i). Conjecture 3.23 was motivated by the author's conjecture (cf. [S14, A233544]) that any integer  $n > 1$  can be written as  $k^2 + m$  with  $\sigma(k^2) + \varphi(m)$  prime, where  $k$  and  $m$  are positive integers with  $m \geq k^2$ , and  $\sigma(j)$  is the sum of all (positive) divisors of  $j$ .

**Conjecture 3.24** (2014-02-01). (i) *For any integer  $n > 13$ , there is a prime  $q < n$  such that  $q + 2$  and  $p_{n-q} + q + 1$  are both prime.*

(ii) *If a positive integer  $n$  is not a divisor of 12, then there is a prime  $q < n$  such that  $3(p_{n-q} + q) - 1$  and  $3(p_{n-q} + q) + 1$  are twin prime.*

*Remark 3.24.* See [S14, A236831 and A182662] for related sequences.

**Conjecture 3.25** (2014-02-26). (i) *For any integer  $n > 10$ , there is a positive integer  $k < n - 1$  such that  $p = p_k + \pi(n - k)$  and  $p + 2$  are both prime.*

(ii) *For any integer  $n > 6$ , there is a positive integer  $k < n$  such that  $p_k + \pi(n - k)$  is a triangular number.*

(iii) *For every  $n = 5, 6, \dots$ , there is a positive integer  $k < n$  such that  $p_k^2 + \pi(n - k)^2$  is prime.*

*Remark 3.25.* See [S14, A238386 and A238405] for related sequences.

## 4. ON PRIMES RELATED TO PARTITION FUNCTIONS

For  $n = 1, 2, 3, \dots$ , let  $p(n)$  denote the number of ways to write  $n$  as a sum of positive integers with the order of addends ignored. The function  $p(n)$  is called the partition function. For each positive integer  $n$ , let  $q(n)$  denote the number of ways to write  $n$  as a sum of *distinct* positive integers with the order of addends ignored. The function  $q(n)$  is usually called the strict partition function. It is known that

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n} \quad \text{and} \quad q(n) \sim \frac{e^{\pi\sqrt{n/3}}}{4(3n^3)^{1/4}} \quad \text{as } n \rightarrow +\infty$$

(cf. [HR] and [AS, p. 826]). So both  $p(n)$  and  $q(n)$  grow eventually faster than any polynomial in  $n$ .

**Conjecture 4.1** (2014-02-27). (i) *Let  $n$  be any positive integer. Then  $p(n) + k$  is prime for some  $k = 1, \dots, n$ . Also,  $q(n) + k$  is prime for some  $k = 1, \dots, n$ .*

(ii) *For each integer  $n > 2$ , there is a positive integer  $k \leq n + 3$  such that  $p(n) - k$  is prime. Also, for any integer  $n > 4$ , there is a number  $k \in \{1, \dots, n\}$  such that  $q(n) + k$  is prime.*

*Remark 4.1.* For example,  $p(5) + 4 = 7 + 4 = 11$  is prime. See [S14, A238457] for the sequence  $a(n) = |\{1 \leq k \leq n : p(n) + k \text{ is prime}\}|$  ( $n = 1, 2, 3, \dots$ ).

**Conjecture 4.2** (2014-02-27). (i) *For any integer  $n > 2$ , there is a prime  $q < n$  with  $2p(n - q) + 1$  prime. Also, for every  $n = 4, 5, \dots$ , there is a prime  $q < n$  with  $2p(n - q) - 1$  prime.*

(ii) *For each integer  $n > 2$ , there is a prime  $p < n$  with  $q(n - p) + 1$  prime. Also, for any integer  $n > 6$ , there is a prime  $p < n$  with  $q(n - p) - 1$  prime.*

*Remark 4.2.* This is an analogue of Conjecture 2.20. We have verified the conjecture for  $n$  up to  $10^5$ . See [S14, A238458 and A238459] for related sequences.

**Conjecture 4.3.** (i) (2013-12-26) *For any integer  $n > 127$ , there is a positive integer  $k < n - 2$  such that  $p(k + \varphi(n - k)/2)$  is prime.*

(ii) (2013-12-28) *For any integer  $n > 727$ , there is a positive integer  $k < n - 2$  such that  $q = \varphi(k) + \varphi(n - k)/2 + 1$  and  $p(q - 1)$  are both prime.*

*Remark 4.3.* Clearly, part (ii) implies that there are infinitely many primes  $q$  with  $p(q - 1)$  prime. We have verified parts (i) and (ii) for  $n$  up to 25000 and 56000 respectively. See [S14, A234470, A234567 and A234569] for related data and graphs.

**Conjecture 4.4** (2013-12-26). *Any integer  $n > 5$  can be written as  $k + m$  with  $k, m \in \{3, 4, \dots\}$  such that  $q(\varphi(k)\varphi(m)/4) + 1$  is prime.*

*Remark 4.4.* This implies that there are infinitely many primes of the form  $q(n) + 1$  with  $n$  a positive integer. See [S14, A234475] for a related sequence.

**Conjecture 4.5** (2013-12-29). *Any integer  $n > 7$  can be written as  $k + m$  with  $k$  and  $m$  positive integers such that  $p = p_k + \varphi(m)$  and  $q(p) - 1$  are both prime. Also, any integer  $n > 7$  not equal to 15 can be written as  $k + m$  with  $k$  and  $m$  positive integers such that  $p = p_k + \varphi(m)$  and  $q(p) + 1$  are both prime.*

*Remark 4.5.* This implies that there are infinitely many primes  $p$  with  $q(p) - 1$  (or  $q(p) + 1$ ) prime. See [S14, A234615 and A234644] for related data and graphs.

**Conjecture 4.6** (2014-01-07). *For any integer  $n \geq 60$ , there is a positive integer  $k < n$  such that  $m \pm 1$  and  $q(m) + 1$  are all prime, where  $m = \varphi(k) + \varphi(n - k)/4$ .*

*Remark 4.6.* This implies that there are infinitely many positive integers  $m$  with  $m \pm 1$  and  $q(m) + 1$  all prime. We have verified the conjecture for  $n$  up to  $10^5$ . See [S14, A235343 and A235344] for related data and graphs.

**Conjecture 4.7** (2014-01-25). (i) *For any integer  $n \geq 128$ , there is a positive integer  $k < n$  such that  $r = \varphi(k) + \varphi(n - k)/6 + 1$  and  $p(r) + q(r)$  are both prime.*

(ii) *For every  $n = 18, 19, \dots$ , there is a positive integer  $k < n$  such that  $m = \varphi(k)/2 + \varphi(n - k)/8$  is an integer with  $p(m)^2 + q(m)^2$  prime.*

*Remark 4.7.* Clearly, part (i) implies that there are infinitely many primes of the form  $p(r) + q(r)$  with  $r$  prime. And part (ii) implies that there are infinitely many positive integers  $m$  with  $p(m)^2 + q(m)^2$  prime. We have verified parts (i) and (ii) for  $n$  up to 30000 and 65000 respectively. See [S14, A236419, A236412 and A236413] for related data and graphs.

For any positive integer  $n$ ,  $\bar{q}(n) = p(n) - q(n)$  is the number of ways to write  $n$  as a sum of unordered positive integers with some part repeated (or even).

**Conjecture 4.8** (2014-01-25). (i) *For any integer  $n \geq 99$ , there is a positive integer  $k < n$  such that  $p = \varphi(k)/2 + \varphi(n - k)/12 + 1$  and  $\bar{q}(p)$  are both prime.*

(ii) *For any integer  $n > 3$ , there is a positive integer  $k < n - 2$  such that  $q(m)^2 + \bar{q}(m)^2$  is prime, where  $m = k + \varphi(n - k)/2$ .*

*Remark 4.8.* Clearly, part (i) implies that there are infinitely many primes of the form  $\bar{q}(p)$  with  $p$  prime. And part (ii) implies that there are infinitely many positive integers  $m$  with  $q(m)^2 + \bar{q}(m)^2$  prime. See [S14, A236417, A236439 and A236440] for related data and graphs.

**Conjecture 4.9** (2014-01-26). (i) *For any integer  $n > 3$ , there is a positive integer  $k < n - 2$  with  $p(k) + 2^{\varphi(n-k)/2}$  prime. Also, any integer  $n > 1$  can be written as  $k + m$  with  $k$  and  $m$  positive integers such that  $2^{\varphi(k)}p(m) + 1$  is prime.*

(ii) Any positive integer  $n$  not among 1, 6, 23, 42 can be written as  $k + m$  with  $k$  and  $m$  positive integers such that  $p(k) + 2p(m)$  is prime.

*Remark 4.9.* See [S14, A236483] for a sequence related to part (i).

**Conjecture 4.10.** (i) (2014-01-26) Any integer  $n > 2$  can be written as  $k + m$  with  $k$  and  $m$  positive integers such that  $q(k) + \bar{q}(m)$  is prime.

(ii) (2013-12-08) For every  $n = 2, 3, \dots$ , there is a positive integer  $k < n$  with  $2^k - 1 + q(n - k)$  prime.

*Remark 4.10.* See [S14, A236442 and A233390] for related data and graphs. We have verified part (ii) for  $n$  up to  $2 \times 10^5$ ; for example, we may take  $k = 17342$  for  $n = 147650$ .

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